On the Benney hierarchy: free energy, string equation and quantization

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Abstract
The bi-Hamiltonian structure of the Benney hierarchy is revisited. We show that the compatibility condition of the Poisson brackets provides the genus-zero free energy of a topological field theory coupled to two-dimensional gravity. We calculate the correlation functions via the Landau–Ginzburg formulation and derive the string equation based on the twistor construction. Moreover, by using the approach of Dubrovin and Zhang, we compute the genus-one correction of the Poisson brackets and compare them with the Oevel–Strampp brackets of the Kaup–Broer hierarchy.

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1. Introduction

Over the past decade, the developments of integrable systems have had many important influences on theoretical physics and pure mathematics. Among others, those works concerning the relationship to topological field theories (TFT) and string theories have had much attention paid to them in frontier subjects (see [1] for a review). In particular, Witten [2] and Kontsevich [3] show that the partition function of two-dimensional (2D) topological gravity is equivalent to a particular tau-function of the Korteweg–de Vries hierarchy characterized by the string equation. Now it is generally believed that 2D TFT coupled to 2D gravity can be formulated as an integrable hierarchy of nonlinear partial differential equations.

In general, 2D TFT can be classified by the solutions of the Witten–Dijkgraaf–E Verlinde–H Verlinde (WDVV) equations of associativity [2, 4] in the sense that a particular solution of WDVV equations provides the primary free energy of some topological model. In fact, various classes of solutions to the WDVV equations have been obtained (see [5–7] and references therein), which turn out to be the tau-functions of dispersionless integrable hierarchies. Accordingly, investigating the solution space of the WDVV equations will deepen our understanding of 2D TFT.

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It is well known that Poisson structures of dispersionless integrable hierarchies have the form of hydrodynamic type [8]. Due to this fact, the integrability (bi-Hamiltonian structure) of the WDVV equations can be formulated in a geometric way called Frobenius manifolds [7]. Based on this geometrical construction, the higher-genus extension of the Poisson structures [9] and Virasoro constraints [10] of the associated integrable hierarchies are given.

The purpose of this paper is to study a dispersionless *non-standard* Lax hierarchy, which is a modification of the dispersionless Kadomtsev–Petviashvili (dKP) hierarchy [11, 12], from the TFT point of view. The Lax operator we would consider has the form

$$L = p^N + v^1 p^{N-1} + v^2 p^{N-2} + \cdots + v^N + \frac{v^{N+1}}{p} \tag{1.1}$$

which satisfies the hierarchy flows ($T_1 = X$)

$$\frac{\partial L}{\partial T_n} = \left\{ L_{\geqslant 1}^{n/N}, L \right\} \tag{1.2}$$

where $L_{\geqslant 1}^n$ denotes the polynomial $p^n + \cdots + (\cdots)p$, i.e. we cut off the terms after p of the expansion (in p) of L^n and the Poisson bracket $\{\,,\,\}$ is defined by [13]

$$\{f(p, X), g(p, X)\} = \frac{\partial f}{\partial p} \frac{\partial g}{\partial X} - \frac{\partial f}{\partial X} \frac{\partial g}{\partial p}.$$

We note that the Lax operator (1.1) can be obtained from that of the dispersionless modified KP (dmKP) hierarchy [14, 15] via truncations. Thus the non-standard Lax hierarchy (1.2) is referred to as the constrained dmKP hierarchy [16].

In this paper, for simplicity, we shall concentrate on the Lax operator of the form

$$L = p + v^1 + v^2 p^{-1} \tag{1.3}$$

which satisfies the non-standard Lax equations

$$\frac{\partial L}{\partial T_n} = \left\{ L_{\geqslant 1}^n, L \right\}. \tag{1.4}$$

The first few flows are

$$\frac{\partial}{\partial T_2} \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = \begin{pmatrix} (v^1)^2 + 2v^2 \\ 2v^1 v^2 \end{pmatrix}_X$$
$$\frac{\partial}{\partial T_3} \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = \begin{pmatrix} 6v^1 v^2 + 6(v^1)^3 \\ 3(v^1)^2 v^2 + 3(v^2)^2 \end{pmatrix}_X \tag{1.5}$$

where the simplest equation (T_2-flow) is the Benney equation which describes long waves in nonlinear phenomena (here we only consider the $(1 + 1)$-dimensional reduction of the Benney system in [17]). The whole equations (1.4) form what we call the Benney hierarchy which has been intensively studied over the past two decades (see, for example, [13, 18–22] and references therein).

Remark 1. The algebraic and Hamiltonian structure associated with the kind of Lax operators (1.1) have also been investigated in [21] where the Lax equations are defined by the bracket $\{A, B\} = p \partial A/\partial p \partial B/\partial x - p \partial A/\partial x \partial B/\partial p$ with respect to the decomposition $\Lambda = \Lambda_{\geqslant 0} \oplus \Lambda_{<0}$ for the pseudo-differential operator Λ and the relationship to the dispersionless Toda hierarchies (dToda) [12, 23] is established as well.

Recently, the bi-Hamiltonian structures associated with the Benney hierarchy (1.4) have been investigated based on the theory of classical r-matrices [14]. From that one can associate with a free energy coming from a 2D TFT. In particular, by this free energy, we construct an additional hierarchy generated by Hamiltonians of logarithmic type which together with the ordinary hierarchy are identified as flows in genus-zero TFT coupled to 2D topological gravity. The basic idea is that the dispersionless Lax operator (1.3) can be viewed as a superpotential in the Landau–Ginzburg (LG) formulation of TFT [4, 5, 24]. Thus, according to the LG theory, the variables v^1 and v^2 are identified as the fundamental correlation functions and their dynamic flows turn out to be the genus-zero topological recursion relations [2, 25] of the associated TFT.

Moreover, in order to establish the string equation describing the gravitational effect, we construct the twistor data for the Benney hierarchy by using the Orlov operator corresponding to the dmKP hierarchy [15]. We show that a remarkable feature shared by the Benney hierarchy is the flows which are generated by the additional logarithmic Hamiltonians can be expressed by the logarithm of the Lax operator and are well defined only after a suitable constraint is included in the twistor data.

Finally, from the genus-zero free energy of the Benney hierarchy, we compute the associated G-function to construct the genus-one free energy and then quantize the Poisson brackets of the Benney hierarchy using the Dubrovin–Zhang (DZ) approach to bi-Hamiltonian structure in 2D TFT [9]. On the other hand, we also 'quantize' these Poisson structures from the Oevel–Strampp (OS) brackets of the Kaup–Broer (KB) hierarchy [26, 27]. We find that after appropriate differential substitutions, they are matched up to genus-one correction.

The paper is organized as follows. In the next section we compute the primary free energy from the bi-Hamiltonian structure of the Benney hierarchy and then introduce the additional logarithmic flows commuting with the ordinary Benney flows. In section 3 we compute topological correlation functions using the LG formulation and show that the Benney hierarchy can be derived from genus-zero topological recursion relations. Section 4 is devoted to finding the twistor data to establish the string equation including the additional logarithmic flows. In section 5 we show that the genus-one correction of the Poisson brackets obtained by DZ quantization coincides with the OS brackets after appropriate differentiable substitutions of the dynamical variables. In the final section we discuss some problems to be investigated.

2. Bi-Hamiltonian structure and free energy

In this section, we shall investigate the relations between the bi-Hamiltonian structure and its associated free energy of the Benney hierarchy.

The bi-Hamiltonian structure of the Benney hierarchy (1.4) is given by [14, 16]

$$\frac{\partial \boldsymbol{v}}{\partial T_n} = J_1 \frac{\delta H_{n+1}}{\delta \boldsymbol{v}} = J_2 \frac{\delta H_n}{\delta \boldsymbol{v}}$$

$$J_1 = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix} \qquad J_2 = \begin{pmatrix} 2\partial & \partial v^1 \\ v^1 \partial & v^2 \partial + \partial v^2 \end{pmatrix} \qquad (2.1)$$

with Hamiltonians defined by

$$H_n = \frac{1}{n} \int \operatorname{res} L^n$$

where $\partial \equiv \partial/\partial X$ and res L^n is the coefficient of p^{-1} of L^n. We list some of them as follows:

$$H_1 = \int v^2$$

$$H_2 = \int v^1 v^2$$

$$H_3 = \int \left[(v^1)^2 v^2 + (v^2)^2 \right]$$

$$H_4 = \int \left[(v^1)^3 v^2 + 3v^1 (v^2)^2 \right].$$

Remark 2. The second bracket J_2 in fact reveals the classical limit of Virasoro–$U(1)$–Kac–Moody algebra [16] with v^2 being the Diff S^1 tensor of weight 2 and v^1 a tensor of weight 1. This is due to the fact that the Diff S^1 flows are just the Hamiltonian flows generated by the Hamiltonian $H_1 = \int v^2$.

Besides the concept of integrability, the geometrical mean of the Poisson brackets (2.1) is profound. The essential idea is based on the fact that the bi-Hamiltonian structure J_1 and J_2 can be written as

$$J_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial \equiv \eta^{\alpha\beta} \partial$$

$$J_2 = \begin{pmatrix} 2 & v^1 \\ v^1 & 2v^2 \end{pmatrix} \partial + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} v^1_X + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} v^2_X \equiv g^{\alpha\beta}(v)\partial + \Gamma^{\alpha\beta}_\gamma(v) v^\gamma_X$$

where $\Gamma^{\alpha\beta}_\gamma(v)$ is the contravariant Levi-Civita connection of the contravariant flat metric $g^{\alpha\beta}(v)$. Both J_1 and J_2 are Poisson brackets of hydrodynamic type introduced by Dubrovin and Novikov [8]. The existence of a bi-Hamiltonian structure means that J_1 and J_2 have to be compatible, i.e. $J = J_1 + \lambda J_2$ must be a Hamiltonian structure as well for all values of λ. The geometric setting of this bi-Hamiltonian structure of hydrodynamic system is provided by Frobenius manifolds [7, 28]. One way to define such manifolds is to construct a function $F(t^1, t^2, \ldots, t^m)$ such that the associated functions,

$$c_{\alpha\beta\gamma} = \frac{\partial^3 F(t)}{\partial t^\alpha \partial t^\beta \partial t^\gamma} \tag{2.2}$$

satisfy the following conditions [7].

- The matrix $\eta_{\alpha\beta} = c_{1\alpha\beta}$ is constant and non-degenerate (for the discussion of degenerate cases, see [29]).
- The functions $c^\alpha_{\beta\gamma} = \eta^{\alpha\epsilon} c_{\epsilon\beta\gamma}$ define an associative commutative algebra with a unity element. The associativity will give a system of nonlinear partial differential equation (PDE) for $F(t)$

$$\frac{\partial^3 F(t)}{\partial t^\alpha \partial t^\beta \partial t^\lambda} \eta^{\lambda\mu} \frac{\partial^3 F(t)}{\partial t^\mu \partial t^\gamma \partial t^\sigma} = \frac{\partial^3 F(t)}{\partial t^\alpha \partial t^\gamma \partial t^\lambda} \eta^{\lambda\mu} \frac{\partial^3 F(t)}{\partial t^\mu \partial t^\beta \partial t^\sigma}. \tag{2.3}$$

- The functions F satisfies a quasi-homogeneity condition, which may be expressed as

$$\mathcal{L}_E F = d_F F + \text{(quadratic terms)}$$

where E is known as the Euler vector field.

Equations (2.3) constitute the WDVV equations [2, 4] arising from TFT (see section 3). A solution of the WDVV equations will be called the primary free energy. Given any solution of the WDVV equation, one can construct a Frobenius manifold $\mathcal{M}$ associated with it. On such a manifold one may interpret $\eta^{\alpha\beta}$ as a flat metric and t^α the flat coordinates. The associativity

can be used to define a Frobenius algebra on each tangent space $T^t \mathcal{M}$. This multiplication will be denoted by $u \cdot v$. Then one may introduce a second flat metric on $\mathcal{M}$ defined by

$$g^{\alpha\beta} = E(dt^\alpha \cdot dt^\beta) \tag{2.4}$$

where $dt^\alpha \cdot dt^\beta = c^{\alpha\beta}_\gamma \, dt^\gamma = \eta^{\alpha\sigma} c^\beta_{\sigma\gamma} \, dt^\gamma$. This metric, together with the original metric $\eta^{\alpha\beta}$, defines a flat pencil (i.e. $\eta^{\alpha\beta} + \lambda g^{\alpha\beta}$ is flat for any value of λ). Thus, one automatically obtains a bi-Hamiltonian structure from a Frobenius manifold $\mathcal{M}$. The corresponding Hamiltonian densities are defined recursively by the formula [7]

$$\frac{\partial^2 h^{(n)}_\alpha}{\partial t^\beta \partial t^\gamma} = c^\sigma_{\beta\gamma} \frac{\partial h^{(n-1)}_\alpha}{\partial t^\sigma} \tag{2.5}$$

where $n \geqslant 1$, $\alpha = 1, 2, \ldots, m$, and $h^{(0)}_\alpha = \eta_{\alpha\beta} t^\beta$. The integrability conditions for this system are automatically satisfied when the $c^\gamma_{\alpha\beta}$ are defined as above.

To find the free energy associated with the Benney hierarchy (1.4), one might set $t^1 = v^1 = h^{(0)}_2, t^2 = v^2 = h^{(0)}_1$ and thus

$$\eta^{\alpha\beta}(t) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad g^{\alpha\beta}(t) = \begin{pmatrix} 2 & t^1 \\ t^1 & 2t^2 \end{pmatrix}. \tag{2.6}$$

We note that the flat metrics $\eta^{\alpha\beta}(t)$ and $g^{\alpha\beta}(t)$ satisfy $\eta^{\alpha\beta}(t) = \partial g^{\alpha\beta}(t)/\partial t^1$ and $\int t^1$ and $\int t^2$ turn out to be the Casimirs for the bi-Hamiltonian structure of the hierarchy. In fact, those $c^\gamma_{\alpha\beta}$ can be determined by (2.5) and (2.6). For $\alpha = 1$ we have

$$\int h^{(n)}_1 = \frac{H_n}{(n-1)!} = \frac{1}{n!} \int \operatorname{res} L^n \qquad (h^{(0)}_1 = t^2)$$

which, up to a normalization, are the Hamiltonian densities of the Benney hierarchy and

$$\begin{aligned}
&c^1_{11} = 1 && c^1_{12} = c^1_{21} = 0 && c^1_{22} = \frac{1}{t^2} \\
&c^2_{11} = c^2_{22} = 0 && c^2_{21} = c^2_{12} = 1.
\end{aligned} \tag{2.7}$$

Then, from (2.7) and (2.2), we immediately obtain the free energy

$$F(t^1, t^2) = \tfrac{1}{2}(t^1)^2 t^2 + \tfrac{1}{2}(t^2)^2 \left(\log t^2 - \tfrac{3}{2}\right). \tag{2.8}$$

Also, from (2.4), (2.6) and (2.7), we can obtain the associated Euler vector field

$$E = t^1 \frac{\partial}{\partial t^1} + 2t^2 \frac{\partial}{\partial t^2}$$

which implies the quasi-homogeneity condition:

$$\mathcal{L}_E F(t) = 4F(t) + (t^2)^2.$$

Next, we turn to the hierarchy corresponding to $\alpha = 2$ with $h^{(0)}_2 = t^1 = v^1$. Using (2.5) and (2.7), we obtain

$$h^{(1)}_2 = \frac{(t^1)^2}{2} + t^2(\log t^2 - 1)$$

$$h^{(2)}_2 = \frac{(t^1)^3}{6} + t^1 t^2(\log t^2 - 1)$$

$$h^{(3)}_2 = \frac{(t^1)^4}{24} + \tfrac{1}{2}(t^1)^2 t^2(\log t^2 - 1) + \tfrac{1}{2}(t^2)^2 \left(\log t^2 - \tfrac{5}{2}\right)$$

$$h^{(4)}_2 = \frac{(t^1)^5}{120} + \tfrac{1}{6}(t^1)^3 t^2(\log t^2 - 1) + \tfrac{1}{2}t^1(t^2)^2 \left(\log t^2 - \tfrac{5}{2}\right).$$

Motivated by the work of [30, 31], the Hamiltonian densities $h_2^{(n)}$ can be expressed as

$$h_2^{(n)} = \frac{2}{n!} \operatorname{res}[L^n(\log L - c_n)]$$

with the prescription

$$\log L = \log(p + t^1 + t^2 p^{-1})$$

$$= \tfrac{1}{2}\log t^2 + \tfrac{1}{2}\log(1 + t^1 p^{-1} + t^2 p^{-2}) + \tfrac{1}{2}\log\left(1 + \frac{t^1}{t^2}p + \frac{1}{t^2}p^2\right) \quad (2.9)$$

and $c_n = \sum_{j=1}^n \frac{1}{j}$, $c_0 = 0$. Then the Lax flows corresponding to $h_2^{(n)}$ are

$$\frac{\partial L}{\partial \bar{T}_n} = 2\{\bar{B}_n, L\} \qquad \bar{B}_n = [L^n(\log L - c_n)]_{\geqslant 1} \quad (2.10)$$

or, in terms of bi-Hamiltonian structure

$$\frac{\partial v}{\partial \bar{T}_n} = J_1 \frac{\delta \bar{H}_{n+1}}{\delta v} = J_2 \frac{\delta \bar{H}_n}{\delta v}$$

where the Hamiltonians $\bar{H}_n$ are defined by

$$\bar{H}_n = \frac{2}{n}\int \operatorname{res}[L^n(\log L - c_n)] = (n-1)! \int h_2^{(n)}.$$

These Hamiltonians generate additional flows which are compatible with the ordinary Benney flows. We will see later that it is these logarithmic flows (2.10) which together with the ordinary flow (1.4) imply that the Benney hierarchy can be formulated as a 2D TFT coupled to gravity.

3. Topological string at genus zero

In this section we would like to set up the correspondence between the Benney hierarchy and its associated TFT at genus zero. Let us first recall some basic notions in TFT.

A topological matter theory can be characterized by a set of BRST invariant observables $\{\mathcal{O}_1, \mathcal{O}_2, \ldots\}$ with couplings $\{T^\alpha\}$ where $\mathcal{O}_1$ denotes the identity operator. If the number of observables is finite the theory is called a topological minimal model and the observables are referred to the primary fields. When the theory couples to gravity, a set of new observables emerge as gravitational descendants $\{\sigma_n(\mathcal{O}_\alpha), n = 1, 2, \ldots\}$ with new coupling constants $\{T^{\alpha,n}\}$. The identity operator $\mathcal{O}_1$ now becomes the puncture operator P. For convenience we can identify the primary fields $\mathcal{O}_\alpha$ and the coupling constants T^α to $\sigma_0(\mathcal{O}_\alpha)$ and $T^{\alpha,0}$, respectively. As usual, we shall call the space spanned by $\{T^{\alpha,n}, n = 0, 1, 2, \ldots\}$ the full phase space and the subspace parametrized by $\{T^\alpha\}$ the small phase space. These coupling times describe the perturbative flows with respect to the corresponding critical theory (in which $T^{\alpha,n} = 0$).

For a topological model the most important quantities are correlation functions which describe the topological properties of the manifold where the model lives. The generating function of correlation functions is the full free energy defined by

$$\mathcal{F}(T) = \sum_{g=0}^\infty \mathcal{F}_g(T) = \sum_{g=0}^\infty \left\langle \exp\left[\sum_{\alpha,n} T^{\alpha,n}\sigma_n(\mathcal{O}_\alpha)\right]\right\rangle_g$$

where $\langle \cdots \rangle_g$ denotes the expectation value on a Riemann surface of genus g with respect to a classical action. In the subsequent sections, we will omit the exponential factor without

causing any confusion. Therefore, a generic m-point correlation function can be calculated as follows:

$$\langle \sigma_{n_1}(\mathcal{O}_{\alpha_1})\sigma_{n_2}(\mathcal{O}_{\alpha_2}) \cdots \sigma_{n_m}(\mathcal{O}_{\alpha_m}) \rangle_g = \frac{\partial^m \mathcal{F}_g}{\partial T^{\alpha_1,n_1} \partial T^{\alpha_2,n_2} \cdots \partial T^{\alpha_m,n_m}}.$$

In the following, we shall restrict ourselves to the trivial topology, i.e. the genus-zero sector ($g = 0$) since this part is more relevant to dispersionless integrable hierarchies. In particular, the genus-zero free energy restricting on the small phase space is the primary free energy defined by

$$\mathcal{F}_0|_{T^\alpha=t^\alpha, T^{\alpha,n\geqslant 1}=0} = F(t).$$

Let us define some genus-zero correlation functions on the small phase space. The metric on the space of primary fields is defined by

$$\langle P\mathcal{O}_\alpha \mathcal{O}_\beta \rangle = \eta_{\alpha\beta}.$$

When $\eta_{\alpha\beta}$ is independent of the couplings we call it the flat metric and the couplings T^α the flat coordinates. In fact, a three-point function in the small phase space can be expressed as

$$\langle \mathcal{O}_\alpha \mathcal{O}_\beta \mathcal{O}\gamma \rangle = \frac{\partial F}{\partial T^\alpha \partial T^\beta \partial T^\gamma} \equiv c_{\alpha\beta\gamma}$$

which provide the structure constants of the commutative associative algebra

$$\mathcal{O}_\alpha \mathcal{O}_\beta = c_{\alpha\beta}^\gamma \mathcal{O}_\gamma$$

with constraints $c_{\alpha\beta}^\gamma = \eta^{\gamma\sigma} c_{\sigma\alpha\beta}$, $c_{1\alpha\beta} = \eta_{\alpha\beta}$. The associativity of $c_{\alpha\beta}^\gamma$, i.e.

$$c_{\alpha\beta}^\mu c_{\mu\gamma}^\sigma = c_{\alpha\gamma}^\mu c_{\mu\beta}^\sigma$$

will give the WDVV equations (2.3).

Now, let us return to the Benney hierarchy. Since the Benney hierarchy is a two-variable theory, only two primary fields $\{\mathcal{O}_1 = \mathcal{O}_P \equiv P, \mathcal{O}_2 = \mathcal{O}_Q \equiv Q\}$ are involved in the TFT formulation and we shall identify $v^1|_{T^{\alpha,n\geqslant 1}=0} = T^P$ and $v^2|_{T^{\alpha,n\geqslant 1}=0} = T^Q$ on the small phase space. Therefore, the Lax operator in small phase space is written as $L(z) = z + T^P + T^Q z^{-1}$ which can be viewed as a superpotential in the LG formulation of TFT [4, 24]. According to the LG theory, the primary fields are defined by

$$\mathcal{O}_P(z) = \frac{\partial L(z)}{\partial T^P} = 1 \qquad \mathcal{O}_Q(z) = \frac{\partial L(z)}{\partial T^Q} = z^{-1}$$

which can be used to compute the three-point correlation functions through the formula [4, 24]:

$$c_{\alpha\beta\gamma} = \text{res}_{L'=0}\left[\frac{\mathcal{O}_\alpha(z)\mathcal{O}_\beta(z)\mathcal{O}_\gamma(z)}{\partial_z L(z)} \right]. \tag{3.1}$$

It is easy to show that (3.1) reproduces the previous $c_{\alpha\beta\gamma}$ and F on the small phase space. In particular, the flat metric on the space of primary fields is given by

$$\eta_{PQ} = \eta_{QP} = 1 \qquad \eta_{PP} = \eta_{QQ} = 0$$

as we obtained previously. Now we can impose the fundamental correlation functions as $\langle PP \rangle = \partial^2 F/\partial(T^P)^2 = T^Q$, $\langle PQ \rangle = \partial^2 F/\partial T^P \partial T^Q = T^P$ and $\langle QQ \rangle = \log T^Q$. Although these two-point correlation functions are defined on the small phase space, however, it has been shown [25] that they can be defined on the full phase space through the variables v^1 and v^2 in which the gravitational couplings $T^{\alpha,n}$ do not vanish. Hence it is easy to write down these

genus-zero two-point functions on the full phase space and obtain the following constitutive relations:

$$\langle PP \rangle = \frac{\partial^2 \mathcal{F}_0}{\partial (T^P)^2} = v^2$$

$$\langle PQ \rangle = \frac{\partial^2 \mathcal{F}_0}{\partial T^P \partial T^Q} = v^1 \tag{3.2}$$

$$\langle QQ \rangle = \log \langle PP \rangle$$

which will be important to provide the connection between the Benney hierarchy and its associated TFT.

Remark 3. In general, for the two-primary models, $\langle QQ \rangle = f(\langle PP \rangle)$ where the function $f(x)$ is model dependent [25]. In [32], the same relation $f(x) = \log x$ has been imposed to extract the nonlinear Schrödinger hierarchy from the Hermitian one-matrix model at finite N. However, for the CP^1 model [2, 25], $f(x) = e^x$.

Based on the constitutive relations (3.2), we can identify the gravitational flows for v^1 and v^2 in the full phase space as the Lax flows by taking into account the genus-zero topological recursion relation [2, 25]:

$$\langle \sigma_n(\mathcal{O}_\alpha) AB \rangle = \sum_{\beta, \gamma = P, Q} n \langle \sigma_{n-1}(\mathcal{O}_\alpha) \mathcal{O}_\beta \rangle \eta^{\beta\gamma} \langle \mathcal{O}_\gamma AB \rangle \qquad \alpha = P, Q. \tag{3.3}$$

For example, setting $n = 1$, $\mathcal{O}_\alpha = P$ and $A = P$, $B = Q$ then

$$\begin{aligned}
\frac{\partial v^1}{\partial T^{P,1}} &= \langle \sigma_1(P) PQ \rangle \\
&= \langle PP \rangle \langle QPQ \rangle + \langle PQ \rangle \langle PPQ \rangle \\
&= \langle PP \rangle \langle QQ \rangle' + \langle PQ \rangle \langle PQ \rangle' \\
&= \left[\tfrac{1}{2}(v^1)^2 + v^2 \right]'
\end{aligned}$$

where we denote $f' = \partial f / \partial T^P = \partial f / \partial X$. Similarly, taking $A = P$, $B = P$ we have

$$\frac{\partial v^2}{\partial T^{P,1}} = \langle \sigma_1(P) PP \rangle = (v^1 v^2)'.$$

On the other hand, taking $n = 2$ we obtain

$$\frac{\partial v^1}{\partial T^{P,2}} = \left[\tfrac{1}{3}(v^1)^3 + 2v^1 v^2 \right]'$$

$$\frac{\partial v^2}{\partial T^{P,2}} = \left[(v^1)^2 v^2 + (v^2)^2 \right]'. \tag{3.4}$$

Comparing the above equations with the Lax flows (1.5), we shall identify $T_n = T^{P,n-1}/n$ ($n = 1, 2, \ldots$).

Next, let us turn to the $T^{Q,n}$ flows. Choosing $\mathcal{O}_\alpha = Q$ and using the topological recursion relation (3.3), we obtain

$$\frac{\partial v^1}{\partial T^{Q,1}} = (v^1 \log v^2)'$$

$$\frac{\partial v^2}{\partial T^{Q,1}} = \left[\tfrac{1}{2}(v^1)^2 + v^2(\log v^2 - 1) \right]'$$

$$\frac{\partial v^1}{\partial T^{Q,2}} = \left[(v^1)^2 \log v^2 + 2v^2(\log v^2 - 2) \right]' \tag{3.5}$$

$$\frac{\partial v^2}{\partial T^{Q,2}} = \left[\tfrac{1}{3}(v^1)^3 + 2v^1 v^2(\log v^2 - 1) \right]'$$

which are nothing but the Lax flows associated with the logarithmic operator $\bar{B}_n$ for $n = 1, 2$ under the identification $\bar{T}_n = T^{Q,n}$.

It turns out that equations (3.4) and (3.5) can be recast in the Lax form in terms of coupling times $T^{\alpha,n}$ as

$$\frac{\partial L}{\partial T^{P,n}} = \frac{1}{n+1}\{L^{n+1}_{\geqslant 1}, L\}$$
$$\frac{\partial L}{\partial T^{Q,n}} = 2\{[L^n(\log L - c_n)]_{\geqslant 1}, L\} \tag{3.6}$$

and the associated commuting Hamiltonian flows with respect to the bi-Hamiltonian structures are thus given by

$$\frac{\partial \boldsymbol{v}}{\partial T^{P,n}} = \{H_{P,n+1}, \boldsymbol{v}\}_1 = \{H_{P,n}, \boldsymbol{v}\}_2 \qquad H_{P,n} = \frac{1}{n(n+1)}\int \operatorname{res} L^{n+1}$$
$$\frac{\partial \boldsymbol{v}}{\partial T^{Q,n}} = \{H_{Q,n+1}, \boldsymbol{v}\}_1 = \{H_{Q,n}, \boldsymbol{v}\}_2 \qquad H_{Q,n} = \frac{2}{n}\int \operatorname{res}[L^n(\log L - c_n)] \tag{3.7}$$

where $n = 0, 1, 2, \dots$.

Remark 4. We note that (2.8) corresponds to the primary free energy of the dKP system in the $W_{0,1}$-model [33] defined by the Lax operator $L = p + v_{-1}/(p - s)$ under the identification $s = T^P$, $v_{-1} = T^Q$. They also show that another choice leads to the dToda system in the CP^1 topological sigma model [30, 31].

Furthermore, using the constitutive relations (3.2) and the Lax flows (3.6), we have

$$\frac{\partial v^1}{\partial T^{\alpha,n}} = \frac{\partial \langle \sigma_n(\mathcal{O}_\alpha)Q\rangle}{\partial T^P} = (R^{(1)}_{\alpha,n})'$$
$$\frac{\partial v^2}{\partial T^{\alpha,n}} = \frac{\partial \langle \sigma_n(\mathcal{O}_\alpha)P\rangle}{\partial T^P} = (R^{(2)}_{\alpha,n})'$$

where $R^{(\beta)}_{\alpha,n}$ are the analogues of the Gel'fand–Dickey potentials [34] of the KP hierarchy, given by

$$R^{(1)}_{P,n} = \langle \sigma_n(P)Q\rangle = \frac{1}{n+1}(L^{n+1})_0$$
$$R^{(1)}_{Q,n} = \langle \sigma_n(Q)Q\rangle = 2[L^n(\log L - c_n)]_0$$
$$R^{(2)}_{P,n} = \langle \sigma_n(P)P\rangle = \frac{1}{n+1}\operatorname{res} L^{n+1} \tag{3.8}$$
$$R^{(2)}_{Q,n} = \langle \sigma_n(Q)P\rangle = 2\operatorname{res}[L^n(\log L - c_n)].$$

For example, the two-point correlators involving the first and the second descendants are

$$\langle \sigma_1(P)P\rangle = v^1 v^2$$
$$\langle \sigma_1(P)Q\rangle = \tfrac{1}{2}(v^1)^2 + v^2$$
$$\langle \sigma_1(Q)P\rangle = \tfrac{1}{2}(v^1)^2 + v^2(\log v^2 - 1)$$
$$\langle \sigma_1(Q)Q\rangle = v^1 \log v^2$$
$$\langle \sigma_2(P)P\rangle = (v^1)^2 v^2 + (v^2)^2$$
$$\langle \sigma_2(P)Q\rangle = \tfrac{1}{3}(v^1)^3 + 2v^1 v^2$$
$$\langle \sigma_2(Q)P\rangle = \tfrac{1}{3}(v^1)^3 + 2v^1 v^2(\log v^2 - 1)$$
$$\langle \sigma_2(Q)Q\rangle = (v^1)^2 \log v^2 + 2v^2(\log v^2 - 2).$$

Finally, we would like to note that the last two equations of (3.8) can be integrated to yield

$$\langle \sigma_n(P) \rangle = \frac{1}{n+1} R^{(2)}_{P,n+1} = \frac{1}{(n+1)(n+2)} \, \text{res} \, L^{n+2}$$

$$\langle \sigma_n(Q) \rangle = \frac{1}{n+1} R^{(2)}_{Q,n+1} = \frac{2}{n+1} \, \text{res}[L^{n+1}(\log L - c_{n+1})]$$

which are just the LG representation for one-point functions of gravitational descendants at genus zero.

4. The twistor data and string equation

In this section we would like to discuss the string equation of the 2D TFT associated with the Benney hierarchy, which govern the dynamics of the variables v^α (or fundamental correlators) in the full phase space. The Lax formulation in section 2 for the Benney hierarchy is, in fact, similar to the formulation of the dToda-type hierarchy [12, 23]. Based on this observation, we can reproduce the Benney equations by imposing constraints on the Lax operators and the associated Orlov operators of the dmKP hierarchy [15] through the twistor data (see below). We shall remark, however, that the flow equations corresponding to $h_2^{(0)}$ are absent in the standard formulation of the dToda-type hierarchy. So we have to properly extend the standard Orlov operator to include the additional hierarchy equations (2.10). We will closely follow that of [35] to show that the constraints imposing on the twistor data implies the string equation of the Benney hierarchy.

Let us consider two Lax operators μ and $\tilde{\mu}$ with the following Laurent expansions $(T^1 = X)$:

$$\mu = p + \sum_{n=0}^{\infty} v^n(T, \tilde{T}) p^{-n}$$

$$\tilde{\mu}^{-1} = \tilde{v}_0(T, \tilde{T}) p^{-1} + \sum_{n=0}^{\infty} \tilde{v}_{n+1}(T, \tilde{T}) p^n \tag{4.1}$$

which satisfy the commuting Lax flows

$$\frac{\partial \mu}{\partial T_n} = \{B_n, \mu\} \qquad \frac{\partial \mu}{\partial \tilde{T}_n} = \{\tilde{B}_n, \mu\}$$

$$\frac{\partial \tilde{\mu}}{\partial T_n} = \{B_n, \tilde{\mu}\} \qquad \frac{\partial \tilde{\mu}}{\partial \tilde{T}_n} = \{\tilde{B}_n, \tilde{\mu}\} \qquad (n = 1, 2, 3, \ldots) \tag{4.2}$$

where the Poisson bracket $\{ , \}$ is defined as before and

$$B_n \equiv (\mu^n)_{\geqslant 1} \qquad \tilde{B}_n \equiv (\tilde{\mu}^{-n})_{\leqslant 0}.$$

Next, we consider the Orlov operators corresponding to dmKP

$$M = \sum_{n=1}^{\infty} n T_n \mu^{n-1} + \sum_{n=1}^{\infty} w_n(T, \tilde{T}) \mu^{-n}$$

$$\tilde{M} = -\sum_{n=1}^{\infty} n \tilde{T}_n \tilde{\mu}^{-n-1} + X + \sum_{n=1}^{\infty} \tilde{w}_n(T, \tilde{T}) \tilde{\mu}^n$$

with the constraint

$$\{\mu, M\} = 1 \qquad \{\tilde{\mu}, \tilde{M}\} = 1. \tag{4.3}$$

In fact, the coefficient functions w_n and $\tilde{w}_n$ in the Orlov operators are defined by the above canonical relations and the following flow equations:

$$\frac{\partial M}{\partial T_n} = \{B_n, M\} \qquad \frac{\partial M}{\partial \tilde{T}_n} = \{\tilde{B}_n, M\}$$

$$\frac{\partial \tilde{M}}{\partial T_n} = \{B_n, \tilde{M}\} \qquad \frac{\partial \tilde{M}}{\partial \tilde{T}_n} = \{\tilde{B}_n, \tilde{M}\} \qquad (n = 1, 2, 3, \ldots). \tag{4.4}$$

Inspired by the twistor construction (or the Riemann–Hilbert problem) for the solution structure of the dToda hierarchy [12, 23], we now give the twistor construction for the Benney hierarchy.

Theorem 1. *Let* $f(p, X), g(p, X), \tilde{f}(p, X), \tilde{g}(p, X)$ *be functions satisfying*

$$\{f(p, X), g(p, X)\} = 1 \qquad \{\tilde{f}(p, X), \tilde{g}(p, X)\} = 1. \tag{4.5}$$

Then the functional equations

$$f(\mu, M) = \tilde{f}(\tilde{\mu}, \tilde{M}) \qquad g(\mu, M) = \tilde{g}(\tilde{\mu}, \tilde{M}) \tag{4.6}$$

can obtain a solution of (4.2) and (4.4). We call the pairs (f, g) *and* $(\tilde{f}, \tilde{g})$ *the twistor data of the solution.*

The proof is provided in appendix A.

To reduce the above theorem to the Benney hierarchy, we have to impose the following constraint on the Lax operators:

$$L = \mu = \tilde{\mu}^{-1} \tag{4.7}$$

that is, $f(p, X) = p$ and $\tilde{f}(p, X) = p^{-1}$. As a result, the time variables $\tilde{T}_n$ can be eliminated via the following identification:

$$\tilde{T}_n = -T_n.$$

From (4.5), the twistor data $g(p, X)$ and $\tilde{g}(p, X)$ can be assumed to be in the following form:

$$g(p, X) = X - \sum_{n=2}^{\infty} nT_n p^{n-1} \qquad \tilde{g}(p, X) = -Xp^2 \tag{4.8}$$

where the second part of $g(p, X)$ is responsible for the string equations (see below). By theorem 1 and equation (4.8), we obtain the following constraint for the Orlov operators:

$$M - \sum_{n=2}^{\infty} nT_n \mu^{n-1} = -\tilde{\mu}^2 \tilde{M}. \tag{4.9}$$

It is the above constraint that leads to the string equations.

So far, the twistor construction only involves the Lax flows (1.4). To obtain the string equations associated with the TFT described in section 2, we have to modify the Orlov operator M to include the additional flows (2.10). Namely, it is necessary to introduce the flows generated by the logarithmic operator

$$\bar{B}_n = [L^n(\log L - c_n)]_{\geqslant 1}$$

where we have imposed the constraint (4.7) and used the prescription of the series expansion (2.9) for $\log L$. Let $\bar{T}_n$ be the time variables of additional flows generated by $\bar{B}_n$ then the Orlov operator M is deformed by these new flows to M' so that

$$\frac{\partial M'}{\partial \bar{T}_n} = 2\{\bar{B}_n, M'\}. \tag{4.10}$$

To construct the modified Orlov operator M', it is convenient to using the dressing method [12]. Let us first express the original Lax operator μ and its conjugate Orlov operator M in dressing form (similarity transformation)

$$\mu = e^{\mathrm{ad}\,\Theta}(p) \qquad M = e^{\mathrm{ad}\,\Theta}\left(\sum_{n=1}^{\infty} nT_n p^{n-1}\right) \tag{4.11}$$

where

$$e^{\mathrm{ad}\,f}(g) = g + \{f, g\} + \frac{1}{2!}\{f, \{f, g\}\} + \cdots.$$

One can understand that this is the canonical transformation generated by $\Theta(T, \bar{T}, p)$ (for the $\bar{T}$ dependence, see below) and its flow equations (Sato equations) can be written as [12]

$$\nabla_{T_n,\Theta}\Theta = B_n - e^{\mathrm{ad}\,\Theta}(p^n) \tag{4.12}$$

where

$$\nabla_{T_n,\Theta}\Theta \equiv \sum_{k=0}^{\infty} \frac{1}{(k+1)!}(\mathrm{ad}\,\Theta)^k\left(\frac{\partial\Theta}{\partial T_n}\right).$$

It is easy to show that (4.12) together with (4.11) implies the flow equations (4.4).

In contrast with equation (4.12), the $\bar{T}_n$ flows for Θ are given by

$$\nabla_{\bar{T}_n,\Theta}\Theta = 2\bar{B}_n - e^{\mathrm{ad}\,\Theta}[p^n(\log p - c_n)]$$

and a similar argument reaches the modified Orlov operator

$$M' = e^{\mathrm{ad}\,\Theta}\left(\sum_{n=1}^{\infty} nT_n p^{n-1} + 2\sum_{n=1}^{\infty} n\bar{T}_n p^{n-1}(\log p - c_{n-1})\right)$$

$$= \sum_{n=1}^{\infty} nT_n \mu^{n-1} + \sum_{n=1}^{\infty} w_n \mu^{-n} + 2\sum_{n=1}^{\infty} n\bar{T}_n \mu^{n-1}(\log \mu - c_{n-1}) \tag{4.13}$$

which satisfies the additional flow equations (4.10).

Now we are in the position to derive the string equation. Following [35] we decompose M' into the positive power part $(M')_{\geqslant 1}$ and the non-positive power part $(M')_{\leqslant 0}$ by using equations (4.9) and (4.13). It turns out that

$$\left(M' - \sum_{n=2}^{\infty} nT_n \mu^{n-1}\right)_{\geqslant 1} = 2\left[\sum_{n=1}^{\infty} n\bar{T}_n \mu^{n-1}(\log \mu - c_{n-1})\right]_{\geqslant 1}$$

$$= 2\sum_{n=1}^{\infty} n\bar{T}_n \bar{B}_{n-1}$$

$$\left(M' - \sum_{n=2}^{\infty} nT_n \mu^{n-1}\right)_{\leqslant 0} = -\left(\tilde{\mu}^2 \tilde{M}\right)_{\leqslant 0}$$

$$= \left[\sum_{n=1}^{\infty} n\tilde{T}_n \tilde{\mu}^{-n+1} - X\tilde{\mu}^2 - \sum_{n=1}^{\infty} \tilde{w}_n \tilde{\mu}^{n+2}\right]_{\leqslant 0}$$

$$= -\left(\sum_{n=1}^{\infty} nT_n \mu^{n-1}\right)_{\leqslant 0}$$

where, by the definition (4.1), we have used the fact that $(\mu^{-n})_{\geqslant 1} = (\tilde{\mu}^n)_{\leqslant 0} = 0$ for $n \geqslant 1$. Hence

$$g(\mu, M') = 2 \sum_{n=1}^{\infty} n \bar{T}_n \bar{B}_{n-1} - \left(\sum_{n=1}^{\infty} n T_n \mu^{n-1} \right)_{\leqslant 0}$$

which together with the canonical commutation relation $\{g(L, M'), L\} = -1$ implies

$$\sum_{n=2}^{\infty} n T_n \frac{\partial L}{\partial T_{n-1}} + \sum_{n=1}^{\infty} n \bar{T}_n \frac{\partial L}{\partial \bar{T}_{n-1}} = -1.$$

After shifting $T_1 \to T_1 - 1$ and making the identification $T_n = T^{P,n-1}/n$ and $\bar{T}_n = T^{Q,n}$ as described before, we have

$$\frac{\partial L}{\partial T^P} = 1 + \sum_{n=1}^{\infty} n T^{P,n} \frac{\partial L}{\partial T^{P,n-1}} + \sum_{n=1}^{\infty} n T^{Q,n} \frac{\partial L}{\partial T^{Q,n-1}}.$$

Now taking the zeroth-order term of the above equation and using the constitutive relation (3.2) yields

$$\frac{\partial \langle P Q \rangle}{\partial T^P} = 1 + \sum_{n=1}^{\infty} \sum_{\alpha=P,Q} n T^{\alpha,n} \langle \sigma_{n-1}(\mathcal{O}_\alpha) P Q \rangle.$$

By integrating the above equation we obtain the universal string equation at genus zero [25]

$$v^1(T) = T^P + \sum_{n=1}^{\infty} \sum_{\alpha=P,Q} n T^{\alpha,n} \langle \sigma_{n-1}(\mathcal{O}_\alpha) Q \rangle$$

$$v^2(T) = T^Q + \sum_{n=1}^{\infty} \sum_{\alpha=P,Q} n T^{\alpha,n} \langle \sigma_{n-1}(\mathcal{O}_\alpha) P \rangle$$

which describe the dynamics of v^α associated with the gravitational background.

5. Genus-one correction of Poisson brackets

In this section, we will compute the genus-one correction to the bi-Hamiltonian structure of the Benney hierarchy by using the DZ approach to the integrable hierarchy associated with TFT [9], which consists of the following two main ingredients:

- introducing slow spatial and time variables scaling

$$T^{\alpha,n} \to \epsilon T^{\alpha,n} \qquad n = 0, 1, 2, \ldots \tag{5.1}$$

- changing the full free energy as

$$\mathcal{F} \to \sum_{g=0}^{\infty} \epsilon^{2g-2} \mathcal{F}_g$$

where ϵ is the parameter of genus expansion. Thus all of the corrections become series in ϵ.

To obtain an unambiguous genus-one correction of the Hamiltonian flows (3.7) one may expand the flat coordinates up to the ϵ^2 order as

$$t_\alpha = t_\alpha^{(0)} + \epsilon^2 t_\alpha^{(1)} + \mathrm{O}(\epsilon^4) \qquad t_\alpha = \eta_{\alpha\beta} t^\beta$$

where $t_\alpha^{(0)}$ are the ordinary Benney variables v_α satisfying (3.7) and $t_\alpha^{(1)}$ is the genus-one correction defined by

$$t_\alpha^{(1)} = \frac{\partial^2 \mathcal{F}_1(T)}{\partial T^\alpha \partial X}.$$

Then there exists a unique hierarchy flows of the form [9]

$$\frac{\partial t^\alpha}{\partial T^{\beta,n}} = \{t^\alpha(X), H_{\beta,n+1}\}_1 + \mathrm{O}(\epsilon^4)$$

$$= \{t^\alpha(X), H_{\beta,n}\}_2 + \mathrm{O}(\epsilon^4) \tag{5.2}$$

with

$$\{t^\alpha(X), t^\beta(Y)\}_i = \{t^\alpha(X), t^\beta(Y)\}_i^{(0)} + \epsilon^2 \{t^\alpha(X), t^\beta(Y)\}_i^{(1)} + \mathrm{O}(\epsilon^4) \qquad i = 1, 2$$

$$H_{\alpha,n} = H_{\alpha,n}^{(0)} + \epsilon^2 H_{\alpha,n}^{(1)} + \mathrm{O}(\epsilon^4).$$

This means that under such a correction the Poisson brackets J_1 and J_2 and the Hamiltonians will receive corrections up to ϵ^2 such that the Hamiltonian flows (5.2) still commute with each other.

In [9], the genus-one part of the free energy has the form

$$\mathcal{F}_1(T) = \left[\tfrac{1}{24} \log \det M_\beta^\alpha(t, \partial_X t) + G(t) \right]_{t=v(T)} \tag{5.3}$$

where the matrix M_β^α is given by

$$M_\beta^\alpha(t, \partial_X t) = c_{\beta\gamma}^\alpha(t) \partial_X t^\gamma$$

and $G(t)$ is a certain function satisfying Getzler's equation [36]. The first part of formula (5.3) is quite simple on the small phase space, whereas the second part describes, in the topological sigma-models, the genus-one Gromov–Witten invariant of the target space and satisfies a complicated recursion relation [36]. For the primary free energy (2.8), the G-function satisfies the following simple ordinary differential equation [9]:

$$48(t^2)^2 \frac{\partial^2 G}{\partial (t^2)^2} + 24 t^2 \frac{\partial G}{\partial t^2} = 2$$

which can be easily solved as (up to a constant)

$$G(t) = -\tfrac{1}{12} \log t^2. \tag{5.4}$$

The G-function of the Frobenius manifold also satisfies the quasi-homogeneity condition $\mathcal{L}_E G = -\tfrac{1}{6}$. The above G-function can also be derived using the tau function of the isomonodromy deformation problem arising in the theory of WDVV equations of associativity [9]. We briefly describe the derivation in appendix B.

On the other hand, a simple computation yields

$$M_\beta^\alpha = \begin{pmatrix} t_X^1 & t_X^2 \\ t_X^2/t^2 & t_X^1 \end{pmatrix}$$

which together with (5.4) implies

$$\mathcal{F}_1 = \tfrac{1}{24} \log \left[(t_X^1)^2 t^2 - (t_X^2)^2 \right] - \tfrac{1}{8} \log t^2.$$

Using $c^{\alpha}_{\beta\gamma}$ and $\mathcal{F}_1$ and consulting the procedure developed by Dubrovin and Zhang (cf theorems 1 and 2 and proposition 3 in [9]), after a straightforward but tedious computation, we obtain the genus-one correction of the first Poisson bracket:

$$\{t^1(X), t^1(Y)\}_1 = 0 + \mathrm{O}(\epsilon^4)$$

$$\{t^1(X), t^2(Y)\}_1 = \delta'(X - Y) + \frac{\epsilon^2}{6}\left[\frac{t_X^2}{(t^2)^2}\delta''(X - Y) - \frac{1}{t^2}\delta'''(X - Y)\right] + \mathrm{O}(\epsilon^4)$$

$$\{t^2(X), t^1(Y)\}_1 = \delta'(X - Y) + \frac{\epsilon^2}{6}\left[\left(\frac{t_{XX}^2}{(t^2)^2} - \frac{2(t_X^2)^2}{(t^2)^3}\right)\delta'(X - Y)\right. \tag{5.5}$$

$$\left. +\frac{2t_X^2}{(t^2)^2}\delta''(X - Y) - \frac{1}{t^2}\delta'''(X - Y)\right] + \mathrm{O}(\epsilon^4)$$

$$\{t^2(X), t^2(Y)\}_1 = 0 + \mathrm{O}(\epsilon^4).$$

On the other hand, for the second bracket, we obtain

$$\{t^1(X), t^1(Y)\}_2 = 2\delta'(X - Y) + \frac{\epsilon^2}{12}\left[\left(\frac{t_{XX}^2}{(t^2)^2} - \frac{2(t_X^2)^2}{(t^2)^3}\right)\delta'(X - Y)\right.$$

$$\left. + \frac{3t_X^2}{(t^2)^2}\delta''(X - Y) - \frac{2}{t^2}\delta'''(X - Y)\right] + \mathrm{O}(\epsilon^4)$$

$$\{t^1(X), t^2(Y)\}_2 = t_X^1\delta(X - Y) + t^1\delta'(X - Y)$$

$$+\frac{\epsilon^2}{6}\left[\left(\frac{t^1 t_X^2}{(t^2)^2} - \frac{t_X^1}{t^2}\right)\delta''(X - Y) - \frac{t^1}{t^2}\delta'''(X - Y)\right] + \mathrm{O}(\epsilon^4) \tag{5.6}$$

$$\{t^2(X), t^1(Y)\}_2 = t^1\delta'(X - Y) + \frac{\epsilon^2}{6}\left[\left(\frac{t^1 t_{XX}^2}{(t^2)^2} + \frac{2t_X^1 t_X^2}{(t^2)^2} - \frac{2t^1(t_X^2)^2}{(t^2)^3} - \frac{t_{XX}^1}{t^2}\right)\delta'(X - Y)\right.$$

$$\left. + \left(\frac{2t^1 t_X^2}{(t^2)^2} - \frac{2t_X^1}{t^2}\right)\delta''(X - Y) - \frac{t^1}{t^2}\delta'''(X - Y)\right] + \mathrm{O}(\epsilon^4)$$

$$\{t^2(X), t^2(Y)\}_2 = t_X^2\delta(X - Y) + 2t^2\delta'(X - Y) + \mathrm{O}(\epsilon^4).$$

Also, we can derive the genus-one corrections of the Hamiltonians, $H_{\alpha,n}^{(1)}$ and some of them are:

$$H_{P,1}^{(1)} = -\int \left(\frac{t_X^1 t_X^2}{6t^2}\right)\mathrm{d}X$$

$$H_{P,2}^{(1)} = -\int \left[\frac{(t_X^1)^2}{6} + \frac{t^1 t_X^1 t_X^2}{6t^2} + \frac{(t_X^2)^2}{8t^2}\right]\mathrm{d}X$$

$$H_{Q,1}^{(1)} = -\int \left[\frac{(t_X^1)^2}{12t^2} + \frac{(t_X^2)^2}{24(t^2)^2}\right]\mathrm{d}X$$

$$H_{Q,2}^{(1)} = -\int \left[\frac{t^1(t_X^1)^2}{12t^2} + \left(\frac{1}{6t^2} + \frac{\log t^2}{6t^2}\right)t_X^1 t_X^2 + \frac{t^1(t_X^2)^2}{24(t^2)^2}\right]\mathrm{d}X.$$

Therefore, the corresponding commuting flow equations up to genus-one corrections are given by

$$\begin{pmatrix} t^1 \\ t^2 \end{pmatrix}_{T^{P,0}} = \begin{pmatrix} t^1 \\ t^2 \end{pmatrix}_X + O(\epsilon^4)$$

$$\begin{pmatrix} t^1 \\ t^2 \end{pmatrix}_{T^{P,1}} = \begin{pmatrix} (t^1)^2/2 + t^2 \\ t^1 t^2 \end{pmatrix}_X + \frac{\epsilon^2}{24} \begin{pmatrix} \dfrac{2t^2_{XX}}{t^2} - \dfrac{3(t^2_X)^2}{(t^2)^2} \\[3mm] 4t^1_{XX} - \dfrac{4t^1_X t^2_X}{t^2} \end{pmatrix}_X + O(\epsilon^4)$$

$$\begin{pmatrix} t^1 \\ t^2 \end{pmatrix}_{T^{Q,0}} = \begin{pmatrix} \log t^2 \\ t^1 \end{pmatrix}_X + \frac{\epsilon^2}{24} \begin{pmatrix} -\dfrac{2t^2_{XX}}{(t^2)^2} + \dfrac{2(t^1_X)^2}{(t^2)^2} + \dfrac{2(t^2_X)^2}{(t^2)^3} \\[3mm] 0 \end{pmatrix}_X + O(\epsilon^4)$$

$$\begin{pmatrix} t^1 \\ t^2 \end{pmatrix}_{T^{Q,1}} = \begin{pmatrix} t^1 \log t^2 \\ \frac{1}{2}(t^1)^2 + t^2(\log t^2 - 1) \end{pmatrix}_X$$

$$+ \frac{\epsilon^2}{24} \begin{pmatrix} \dfrac{4t^1_{XX}}{t^2} - \dfrac{2t^1 t^2_{XX}}{(t^2)^2} + \dfrac{2t^1(t^1_X)^2}{(t^2)^2} + \dfrac{2t^1(t^2_X)^2}{(t^2)^3} - \dfrac{6t^1_X t^2_X}{(t^2)^2} \\[3mm] \dfrac{4t^2_{XX}}{t^2} - \dfrac{2(t^1_X)^2}{t^2} - \dfrac{5(t^2_X)^2}{(t^2)^2} \end{pmatrix}_X + O(\epsilon^4).$$

Finally, we would like to show that the genus-one correction of the Poisson brackets (5.5) and (5.6) can be rederived from the 'quantum' brackets associated with a dispersive counterpart of the Benney hierarchy. This means that the loop correction can be viewed as the dispersive effect of the hydrodynamic-type Poisson structure. Let us consider the 'quantum' Lax operator of the form

$$K = \epsilon \partial + u^1 + (\epsilon \partial)^{-1} u^2 \qquad \partial \equiv \partial/\partial X$$

which is just the Lax operator of the KB hierarchy discussed in [26, 27] under the scaling (5.1). The bi-Hamiltonian structure associated with K has been obtained by Oevel and Strampp [27] as follows:

$$\{I, J\}_i = \int \text{res} \left[\frac{\delta I}{\delta K} \Omega_i \left(\frac{\delta J}{\delta K} \right) \right] \qquad i = 1, 2$$

where I and J are functionals of K and the Hamiltonian maps Ω_i are defined by

$$\Omega_1 : \frac{\delta I}{\delta K} \to \left[\left(\frac{\delta I}{\delta K} \right)_{\geqslant 1}, K \right] - \left(\left[\frac{\delta I}{\delta K}, K \right] \right)_{\geqslant -1}$$

$$\Omega_2 : \frac{\delta I}{\delta K} \to \left(K \frac{\delta I}{\delta K} \right)_+ K - K \left(\frac{\delta I}{\delta K} K \right)_+ - \left[\left(K \frac{\delta I}{\delta K} \right)_0, K \right] - \left(\left[\frac{\delta I}{\delta K}, K \right] \right)_{-1} K$$

$$+ \left[\int^X \text{res} \left[\frac{\delta I}{\delta K}, K \right], K \right]$$

with

$$\frac{\delta I}{\delta K} = \frac{\delta I}{\delta u^2} + (\epsilon \partial)^{-1} \frac{\delta I}{\delta u^1}.$$

Using the Hamiltonian flows $\partial K/\partial t = \Theta_i(\delta H/\delta K)$, we can easily read off the 'quantum' Poisson brackets

$$\{u^1(X), u^1(Y)\}_1 = 0$$

$$\{u^1(X), u^2(Y)\}_1 = \delta'(X - Y)$$

$$\{u^2(X), u^1(Y)\}_1 = \delta'(X - Y)$$

$$\{u^2(X), u^2(Y)\}_1 = 0$$

for the first structure and

$$\{u^1(X), u^1(Y)\}_2 = 2\delta'(X - Y)$$

$$\{u^1(X), u^2(Y)\}_2 = u^1_X \delta(X - Y) + u^1 \delta'(X - Y) + \epsilon \delta''(X - Y)$$

$$\{u^2(X), u^1(Y)\}_2 = u^1 \delta'(X - Y) - \epsilon \delta''(X - Y)$$

$$\{u^2(X), u^2(Y)\}_2 = u^2_X \delta(X - Y) + 2u^2 \delta'(X - Y)$$

for the second. As a result, the first structure gets no correction, whereas the second structure receives a first-order correction. So far, everything is exact. However, if we define the following substitution for the flat coordinates t^α:

$$t^1(T) = u^1 - \epsilon(\ln u^2)_X - \frac{\epsilon^2}{24}\left(\frac{u^1_X}{u^2}\right)_X + \frac{\epsilon^3}{72}\left[\frac{(\ln u^2)_{XX}u^2 - (u^1_X)^2}{(u^2)^2}\right]_X + O(\epsilon^4)$$

$$t^2(T) = u^2 - \frac{\epsilon}{2}u^1_X + \frac{3\epsilon^2}{8}(\ln u^2)_{XX} + \frac{11\epsilon^3}{144}\left(\frac{u^1_X}{u^2}\right)_{XX} + O(\epsilon^4) \tag{5.7}$$

where the right-hand side of t^α is constructed from $\partial^i v^\alpha / \partial (T^Q)^i|_{v^\alpha = u^\alpha}$, then a straightforward but lengthy calculation shows that the first and the second Poisson brackets for t^α coincide with equations (5.5) and (5.6) modulo $O(\epsilon^4)$. Furthermore, using (5.7), it is not hard to check that the dispersion expansion of the Hamiltonians and hierarchy flows defined by the KB hierarchy and those defined by $H_{P,n}$ and $T^{P,n}$-flow coincide, modulo $O(\epsilon^4)$. In this sense, the parameter ϵ of genus expansion characterizes the effect of dispersion.

Remark 5. The free energy and G-function associated with the bi-Hamiltonian structure of Drinfeld–Sokolov reduction of Lie algebra B_2 are [9]

$$F = \tfrac{1}{2}(t^1)^2 t^2 + \tfrac{1}{15}(t^2)^5 \qquad G = -\tfrac{1}{48}\log t^2$$

where the G-function of B_2 is also of logarithmic type. However, in contrast with the Benney hierarchy, the second Poisson brackets of the dispersion expansion coincide with those of DZ brackets only up to ϵ^0 [9]. This inconsistency was also pointed out in [37] by considering the commuting flows.

6. Conclusions

We have studied several interesting properties associated with the bi-Hamiltonian structure of the Benney hierarchy. Starting with the Poisson brackets of hydrodynamic type we obtain the structure coefficients of an associative algebra characterizing the associated Frobenius manifold. This implies that there exists a function which generates the structure coefficients and this function, in fact, can be viewed as the genus-zero primary free energy of a TFT. It turns out that the topological correlation functions at genus zero can be constructed by using the LG formulation. After appropriately defining the two-point correlation functions as the Benney

variables, the genus-zero topological recursion relation of the TFT turns out to be the Benney equations. Furthermore, we use the twistor construction to derive the string equation associated with the TFT, which describe the dynamics of correlation functions in the full phase space. Finally, based on the approach of Dubrovin and Zhang we obtain the genus-one correction of the Poisson brackets. We show that the same result can be reached by analysing the OS brackets of the KB hierarchy in the semiclassical limit.

In spite of the results obtained, there are some interesting issues deserving more investigations.

- One knows that there exists a Legendre-type transformation between the free energy (2.8) and that of topological CP^1 model [31]. In fact, one can check that the DZ brackets of the Benney hierarchy and those of dToda system in the CP^1 model [9, 37] coincide only up to ϵ^0 via the Legendre-type transformation. A natural question is: can we find a 'quantized' version of the Legendre transformation between these two dispersionless hierarchies so that their DZ brackets match up to $g \geqslant 1$?
- In the deviation of the string equation we borrow the method of twistor construction to obtain the result. In [10], using a recursion procedure, Dubrovin and Zhang establish the Virasoro constraints of genus zero for an arbitrary Frobenius manifold. Then it would be interesting to know whether the solution of the string equation of the Benney hierarchy satisfies Virasoro constraints using the methods developed in [5, 23]. Also, we wonder whether there exists a matrix model associated with the Benney hierarchy such that the large-N limit of that reproduces the genus expansion of the hierarchy flows.
- Recently, the genus-two free energy $\mathcal{F}_2$ of the Benney hierarchy has been obtained in [38] (see also [39]) by combining genus-two topological recursion relations and Virasoro constraints [40]. So we should extend the expansions (5.5) and (5.6) to $O(\epsilon^6)$ when the genus-two correction is included. It is quite interesting to see whether the expansions (5.7) can be extended so that the OS brackets are matched with the DZ brackets up to $O(\epsilon^6)$ after appropriate differential substitutions.

We hope to report on these issues in a forthcoming paper.

Acknowledgments

Appendix A. Proof of theorem 1

Let us first derive the canonical Poisson relations (4.3). By the chain rule, we have

$$
\begin{pmatrix} \dfrac{\partial f(\mu, M)}{\partial \mu} & \dfrac{\partial f(\mu, M)}{\partial M} \\[2mm] \dfrac{\partial g(\mu, M)}{\partial \mu} & \dfrac{\partial g(\mu, M)}{\partial M} \end{pmatrix} \begin{pmatrix} \dfrac{\partial \mu}{\partial p} & \dfrac{\partial \mu}{\partial X} \\[2mm] \dfrac{\partial M}{\partial p} & \dfrac{\partial M}{\partial X} \end{pmatrix}
$$
$$
= \begin{pmatrix} \dfrac{\partial \tilde{f}(\tilde{\mu}, \tilde{M})}{\partial \tilde{\mu}} & \dfrac{\partial \tilde{f}(\tilde{\mu}, \tilde{M})}{\partial \tilde{M}} \\[2mm] \dfrac{\partial \tilde{g}(\tilde{\mu}, \tilde{M})}{\partial \tilde{\mu}} & \dfrac{\partial \tilde{g}(\tilde{\mu}, \tilde{M})}{\partial \tilde{M}} \end{pmatrix} \begin{pmatrix} \dfrac{\partial \tilde{\mu}}{\partial p} & \dfrac{\partial \tilde{\mu}}{\partial X} \\[2mm] \dfrac{\partial \tilde{M}}{\partial p} & \dfrac{\partial \tilde{M}}{\partial X} \end{pmatrix}. \tag{A.1}
$$

Taking the determinant of both hand sides of this equation and using relations (4.5), we obtain

$$\{\mu, M\} = \{\tilde{\mu}, \tilde{M}\}. \tag{A.2}$$

One can calculate the left-hand side as

$$
\begin{aligned}
\{\mu, M\} &= \frac{\partial \mu}{\partial p}\frac{\partial M}{\partial X} - \frac{\partial M}{\partial p}\frac{\partial \mu}{\partial X} \\
&= \frac{\partial \mu}{\partial p}\left[\left(\frac{\partial M}{\partial \mu}\right)_{w_n(T,\tilde{T})\text{ fixed}}\frac{\partial \mu}{\partial X} + 1 + \sum_{i=1}^{\infty}\frac{\partial w_i(T,\tilde{T})}{\partial X}\mu^{-i}\right] \\
&\quad - \frac{\partial \mu}{\partial X}\left(\frac{\partial M}{\partial \mu}\right)_{w_n(T,\tilde{T})\text{ fixed}}\frac{\partial \mu}{\partial p} \\
&= 1 + (\text{negative powers of } p)
\end{aligned}
$$

where we have used the fact that the terms containing $\left(\frac{\partial M}{\partial \mu}\right)_{w_n(T,\tilde{T})\text{ fixed}}$ in the last line cancel. Similar calculations can show that the right-hand side contains only the non-negative powers of p. Therefore, the both hands of (A.2) are equal to 1, which proves the canonical relations (4.3).

The Lax equations with respect to T_n are proved as follows. Differentiating equations (4.6) by T_n gives

$$
\begin{pmatrix} \dfrac{\partial f(\mu, M)}{\partial \mu} & \dfrac{\partial f(\mu, M)}{\partial M} \\[2ex] \dfrac{\partial g(\mu, M)}{\partial \mu} & \dfrac{\partial g(\mu, M)}{\partial M} \end{pmatrix}\begin{pmatrix} \dfrac{\partial \mu}{\partial T_n} \\[2ex] \dfrac{\partial M}{\partial T_n} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial \tilde{f}(\tilde{\mu}, \tilde{M})}{\partial \tilde{\mu}} & \dfrac{\partial \tilde{f}(\tilde{\mu}, \tilde{M})}{\partial \tilde{M}} \\[2ex] \dfrac{\partial \tilde{g}(\tilde{\mu}, \tilde{M})}{\partial \tilde{\mu}} & \dfrac{\partial \tilde{g}(\tilde{\mu}, \tilde{M})}{\partial \tilde{M}} \end{pmatrix}\begin{pmatrix} \dfrac{\partial \tilde{\mu}}{\partial T_n} \\[2ex] \dfrac{\partial \tilde{M}}{\partial T_n} \end{pmatrix}. \tag{A.3}
$$

Using (A.1), we can rewrite (A.3) as

$$
\begin{pmatrix} \dfrac{\partial \mu}{\partial p} & \dfrac{\partial \mu}{\partial X} \\[2ex] \dfrac{\partial M}{\partial p} & \dfrac{\partial M}{\partial X} \end{pmatrix}^{-1}\begin{pmatrix} \dfrac{\partial \mu}{\partial T_n} \\[2ex] \dfrac{\partial M}{\partial T_n} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial \tilde{\mu}}{\partial p} & \dfrac{\partial \tilde{\mu}}{\partial X} \\[2ex] \dfrac{\partial \tilde{M}}{\partial p} & \dfrac{\partial \tilde{M}}{\partial X} \end{pmatrix}^{-1}\begin{pmatrix} \dfrac{\partial \tilde{\mu}}{\partial T_n} \\[2ex] \dfrac{\partial \tilde{M}}{\partial T_n} \end{pmatrix}. \tag{A.4}
$$

Since the determinants of the 2×2 matrices on both sides are 1, the inverse can also be written explicitly. In components, thus, the above matrix (A.4) relation gives

$$\frac{\partial M}{\partial X}\frac{\partial \mu}{\partial T_n} - \frac{\partial \mu}{\partial X}\frac{\partial M}{\partial T_n} = \frac{\partial \tilde{M}}{\partial X}\frac{\partial \tilde{\mu}}{\partial T_n} - \frac{\partial \tilde{\mu}}{\partial X}\frac{\partial \tilde{M}}{\partial T_n} \tag{A.5}$$

$$\frac{\partial M}{\partial p}\frac{\partial \mu}{\partial T_n} - \frac{\partial \mu}{\partial p}\frac{\partial M}{\partial T_n} = \frac{\partial \tilde{M}}{\partial p}\frac{\partial \tilde{\mu}}{\partial T_n} - \frac{\partial \tilde{\mu}}{\partial p}\frac{\partial \tilde{M}}{\partial T_n}. \tag{A.6}$$

The left-hand sides of equations (A.5) and (A.6) can be calculated just as we have done above for derivatives in (p, X). For the equation of (A.5),

$$
\begin{aligned}
\frac{\partial M}{\partial X}\frac{\partial \mu}{\partial T_n} - \frac{\partial \mu}{\partial X}\frac{\partial M}{\partial T_n} &= \left[\left(\frac{\partial M}{\partial \mu}\right)_{w_i(T,\tilde{T})\text{ fixed}}\frac{\partial \mu}{\partial X} + 1 + \sum_{i=1}^{\infty}\frac{\partial w_i(T,\tilde{T})}{\partial X}\mu^{-i}\right]\frac{\partial \mu}{\partial T_n} \\
&\quad - \frac{\partial \mu}{\partial X}\left[\left(\frac{\partial M}{\partial \mu}\right)_{w_i(T,\tilde{T})\text{ fixed}}\frac{\partial \mu}{\partial T_n} + n\mu^{n-1} + \sum_{i=1}^{\infty}\frac{\partial w_i(T,\tilde{T})}{\partial T_n}\mu^{-i}\right]
\end{aligned}
$$

and terms containing $\left(\frac{\partial M}{\partial \mu}\right)_{w_i(T,\tilde{T}) \text{ fixed}}$ cancel. Thus,

$$\frac{\partial M}{\partial X}\frac{\partial \mu}{\partial T_n} - \frac{\partial \mu}{\partial X}\frac{\partial M}{\partial T_n} = -\frac{\partial (\mu^n)_{\geqslant 1}}{\partial X} + (\text{powers of } p \leqslant 0).$$

Similar calculations show that the right-hand side of (A.5) contains only positive powers of p. Therefore, only powers of $p \geqslant 1$ should survive. Hence

$$\frac{\partial M}{\partial X}\frac{\partial \mu}{\partial T_n} - \frac{\partial \mu}{\partial X}\frac{\partial M}{\partial T_n} = -\frac{\partial (\mu^n)_{\geqslant 1}}{\partial X} = -\frac{\partial B_n}{\partial X}. \tag{A.7}$$

For equation (A.6), we have similarly

$$\frac{\partial M}{\partial p}\frac{\partial \mu}{\partial T_n} - \frac{\partial \mu}{\partial p}\frac{\partial M}{\partial T_n} = -\frac{\partial (\mu^n)_{\geqslant 0}}{\partial p} + (\text{negative powers of } p)$$

$$= -\frac{\partial (\mu^n)_{\geqslant 1}}{\partial p} + (\text{negative powers of } p).$$

Similarly, noticing the partial derivative $\frac{\partial}{\partial p}$, we can also show that the right-hand of (A.6) has a Laurent expansion with only non-negative powers of p. Hence only non-negative powers of p should survive. Thus

$$\frac{\partial M}{\partial p}\frac{\partial \mu}{\partial T_n} - \frac{\partial \mu}{\partial p}\frac{\partial M}{\partial T_n} = -\frac{\partial (\mu^n)_{\geqslant 1}}{\partial p} = -\frac{\partial B_n}{\partial p}. \tag{A.8}$$

Using

$$\{\mu, M\} = \{\tilde{\mu}, \tilde{M}\} = 1$$

equations (A.7) and (A.8) can be readily solved:

$$\frac{\partial \mu}{\partial T_n} = -\frac{\partial \mu}{\partial p}\frac{\partial B_n}{\partial X} + \frac{\partial \mu}{\partial X}\frac{\partial B_n}{\partial p} = \{B_n, \mu\}$$

$$\frac{\partial M}{\partial T_n} = -\frac{\partial M}{\partial p}\frac{\partial B_n}{\partial X} + \frac{\partial M}{\partial X}\frac{\partial B_n}{\partial p} = \{B_n, M\}$$

which is nothing but the T_n-flow part of the Lax equations (4.2) and (4.4). The $\tilde{T}_n$-flow part of the Lax equations can be proved in a similar way. This completes the proof of the theorem.

Appendix B. A deviation of the G-function

For the quasi-homogeneous primary free energy (2.8), one can introduce the canonical coordinates s^1, s^2 determined by

$$\det(g^{ij}(t) - s\eta^{ij}(t)) = 0$$

which gives

$$s^1 = t^1 - 2\sqrt{t^2} \qquad s^2 = t^1 + 2\sqrt{t^2}.$$

Then the topological metric η^{ij} and its inverse η_{ij} in the canonical coordinates are given by

$$\eta^{ij}(s) = \frac{\partial s^i}{\partial t^k}\frac{\partial s^j}{\partial t^l}\eta^{kl}(t) = \begin{pmatrix} -\dfrac{8}{s^2 - s^1} & 0 \\ 0 & \dfrac{8}{s^2 - s^1} \end{pmatrix}$$

$$\eta_{ij}(s) = \begin{pmatrix} -\dfrac{s^2 - s^1}{8} & 0 \\ 0 & \dfrac{s^2 - s^1}{8} \end{pmatrix}. \tag{B.1}$$

It has been shown [9] that the G-function can be expressed by the following formula:

$$G(t^1, t^2) = \log \frac{\tau_I}{J^{1/24}}$$

where J is the Jacobian of the transformation from the canonical coordinates to the flat ones, which is easily obtained as

$$J = \det\left(\frac{\partial t^i}{\partial s^j}\right) = \frac{1}{2}\sqrt{t^2}$$

and τ_I is the tau-function of a solution in the theory of isomonodromy deformation defined by the quadrature [41]

$$\mathrm{d}\log\tau_I = \sum_{i=1}^{2} H_i\, \mathrm{d}s^i \tag{B.2}$$

where the quadratic Hamiltonian H_i has the form

$$H_i = \frac{1}{2}\sum_{j\neq i}\frac{V_{ij}^2}{s^i - s^j}$$

with

$$V_{ij} = -(s^i - s^j)\gamma_{ij}(s) \qquad \gamma_{ij}(s) = \frac{\partial_j\sqrt{\eta_{ii}(s)}}{\sqrt{\eta_{jj}(s)}}.$$

Note that $\partial_i \equiv \frac{\partial}{\partial s^i}$ and $V_{ij} = -V_{ji}$.

From (B.1) we have

$$\gamma_{ij}(s) = \begin{pmatrix} \dfrac{1}{2(s^1 - s^2)} & \dfrac{-\mathrm{i}}{2(s^2 - s^1)} \\ \dfrac{-\mathrm{i}}{2(s^2 - s^1)} & \dfrac{1}{2(s^2 - s^1)} \end{pmatrix} \qquad V_{ij} = \begin{pmatrix} 0 & -\frac{1}{2}\mathrm{i} \\ \frac{1}{2}\mathrm{i} & 0 \end{pmatrix}$$

which together with (B.2) implies

$$\tau_I = (s^2 - s^1)^{-1/8}$$

and hence, up to an additive constant

$$G(t^1, t^2) = \log \frac{(s^2 - s^1)^{-1/8}}{[(s^2 - s^1)/8]^{1/24}} = -\frac{1}{12}\log t^2.$$

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